Learning Nonlinear and Deep Representations from High-Dim Data

# Nonconvex Optimization of Low-Dimensional Models

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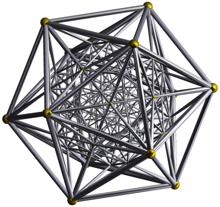


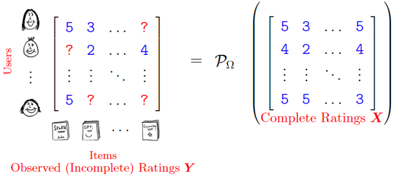
# Outline

1. Introduction & Motivation of Nonconvex Optimization
   - Motivating Examples
   - Nonlinearality, Nonconvexity, and Symmetry
2. Symmetry & Geometry for Nonconvex Problems in Practice
   - Problems with Rotational Symmetry
   - Problems with Discrete Symmetry
3. Efficient Nonconvex Optimization
   - Objectives of Nonconvex Optimization

# Example: Low-rank Matrix Completion

We observe:

$$\underbrace{\boldsymbol{Y}}_{\text{Observed ratings}} = \mathcal{P}_\Omega \Big[ \underbrace{\boldsymbol{X}}_{\text{Complete ratings}} \Big].$$




**Matrix completion**

via bilinear low-rank factorization

$$\min_{\boldsymbol{U},\boldsymbol{V}} f(\boldsymbol{U},\boldsymbol{V}) = \sum_{(i,j)\in\Omega} [(\boldsymbol{U}\boldsymbol{V}^*)_{i,j} - \boldsymbol{Y}_{i,j}]^2 + \underbrace{\frac{\lambda}{2}\|\boldsymbol{U}\|_F^2 + \frac{\lambda}{2}\|\boldsymbol{V}\|_F^2}_{\mathrm{reg}(\boldsymbol{U},\boldsymbol{V})}.$$

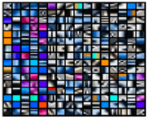
$$\|\boldsymbol{M}\|_* = \min_{\boldsymbol{M}=\boldsymbol{U}\boldsymbol{V}^*} \frac{\lambda}{2}\|\boldsymbol{U}\|_F^2 + \frac{\lambda}{2}\|\boldsymbol{V}\|_F^2$$

# Example: Dictionary for Image Representation

Image processing
(e.g. denoising or super-resolution)
against a known sparsifying dictionary:

$$I_{\text{noisy}} = \underbrace{\boldsymbol{A}}_{\text{dictionary}} \times \underbrace{\boldsymbol{x}}_{\text{sparse}} + \underbrace{\boldsymbol{z}}_{\text{noise}}. \quad (1)$$



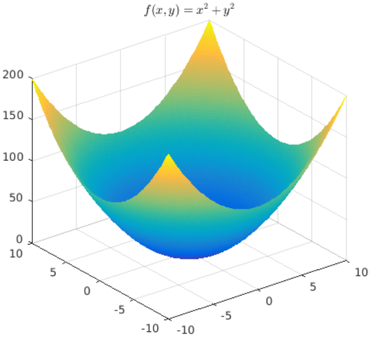


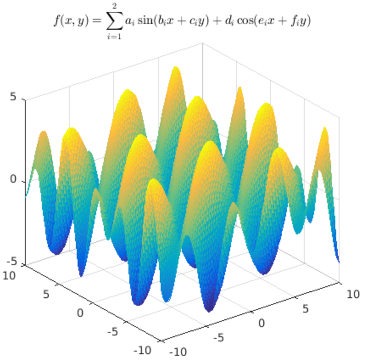
**Dictionary learning**: the motifs or atoms of the dictionary are unknown:

$$\underbrace{\boldsymbol{Y}}_{\text{data}} = \underbrace{\boldsymbol{A}}_{\text{dictionary}} \underbrace{\boldsymbol{X}}_{\text{sparse}}. \quad (2)$$

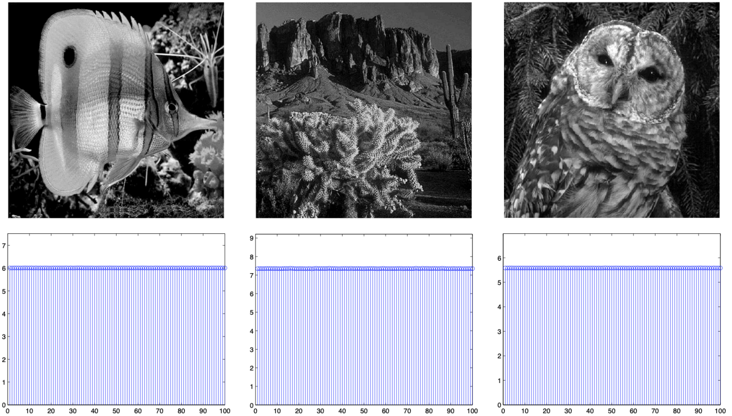
- Band-limited signals: $\boldsymbol{A} = \boldsymbol{F}$, the Fourier transform;
- Piecewise smooth signals: $\boldsymbol{A} = \boldsymbol{W}$, the wavelet transforms;
- Natural images $\boldsymbol{A} =?$ (How to **learn** $\boldsymbol{A}$ from the data $\boldsymbol{Y}$?)

# Convex and Nonconvex Optimization



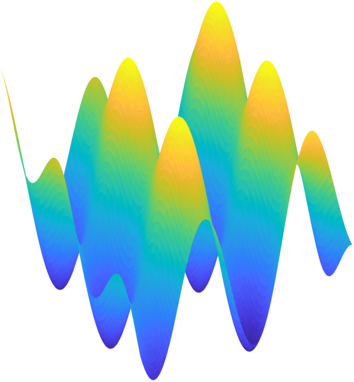



# Dictionary Learning

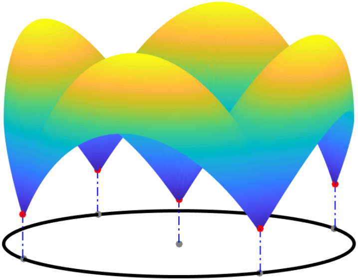



**Recovered solutions always obtain the same objective value.**

# Benign Nonconvex Optimization Landscape



**General Case**



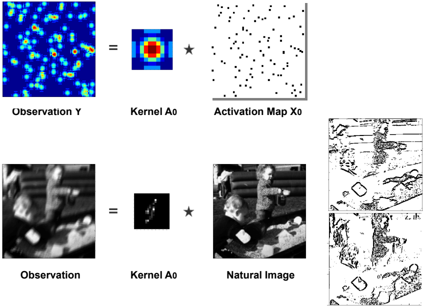
**Structured Case**

# Example: Sparse Blind Deconvolution

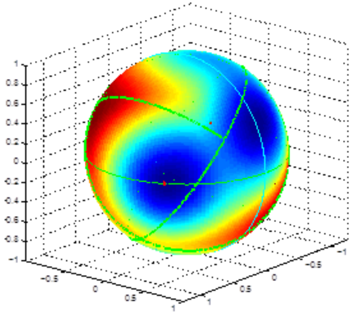
**Sparse Blind Deconvolution**:
the convolutional motif or sparse activation signal are unknown:

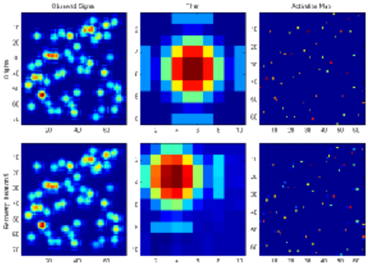
$$\underset{\textbf{data}}{\boldsymbol{Y}} = \underset{\textbf{motif}}{\boldsymbol{A}} * \underset{\textbf{sparse}}{\boldsymbol{X}}. \tag{3}$$

- Scientific signals:
  activation signals are sparse
- Image deblurring:
  natural images are
  sparse in the gradient domain



# Sparse Blind Deconvolution

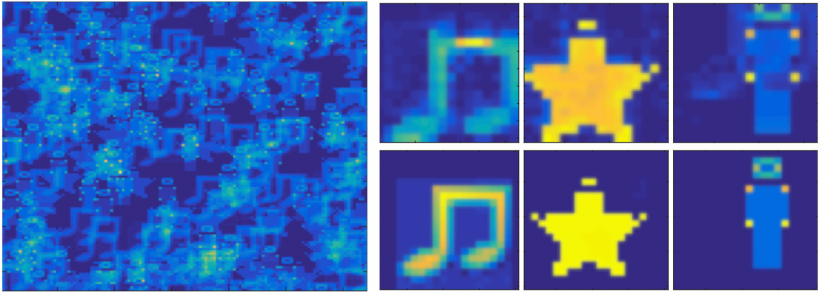


**Recovered solutions are near signed shift-truncations of the ground truth.**

# Convolutional Dictionary learning

$$\underbrace{\boldsymbol{Y}}_{\text{data}} = \sum_i \underbrace{\boldsymbol{A}_i}_{\text{motif}} * \underbrace{\boldsymbol{X}_i}_{\text{sparse}}.$$

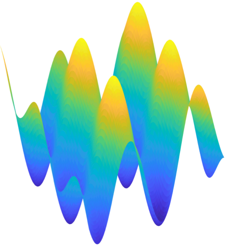


**Recovered solutions are near signed shift-truncations of the ground truth.**

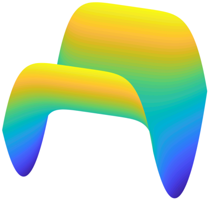
# Challenges of Nonconvex Optimization – Pessimistic Views

Consider the problem of minimizing a general nonlinear function:

$$\min_{\boldsymbol{z}} \varphi(\boldsymbol{z}), \quad \boldsymbol{z} \in \mathsf{C}. \qquad (4)$$



Spurious local minimizers



Flat saddle points

In the worst case, even finding a *local* minimizer can be NP-hard[1].

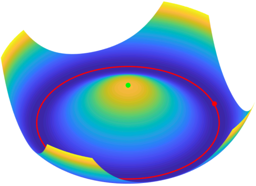
Hence typically people seek to work with relatively benign functions with benign guarantees:

1. convergence to some critical point $\bar{\boldsymbol{z}}$ such that $\nabla\varphi(\bar{\boldsymbol{z}}) = \mathbf{0}$;
2. or convergence to some local minimizer $\nabla^2\varphi(\bar{\boldsymbol{z}}) \succeq \mathbf{0}$.

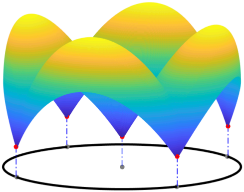
[1] Some NP-complete problems in quadratic and nonlinear programming, K.G Murty and S. N. Kabadi, 1987

# Opportunities – Optimistic Views

However, nonconvex problems that arise from natural physical, geometrical, or statistical origins typically have nice structures, in terms of symmetries!



Rotational symmetry



Discrete symmetry

The function $\varphi$ is invariant under certain group action:

- for low rank matrix recovery, invariant under a continuous rotation:

$$\varphi((\boldsymbol{U}\boldsymbol{\Gamma}, \boldsymbol{V}\boldsymbol{\Gamma}^{-1})) = \varphi((\boldsymbol{U}, \boldsymbol{V})), \quad \forall \text{ invertible } \boldsymbol{\Gamma}.$$

- for dictionary learning, invariant under signed permutations:

$$\varphi((\boldsymbol{A}, \boldsymbol{X})) = \varphi((\boldsymbol{A}\boldsymbol{\Pi}, \boldsymbol{\Pi}^* \boldsymbol{X})), \quad \forall \boldsymbol{\Pi} \in \mathsf{SP}(n).$$

# Nonlinearity and Symmetry

Intrinsic ambiguity against the uniqueness of the solution

- **low rank matrix recovery**

$$\boldsymbol{X} = \boldsymbol{U}_0 \boldsymbol{V}_0^T = \boldsymbol{U}_0 \boldsymbol{\Gamma} \boldsymbol{\Gamma}^{-1} \boldsymbol{V}_0^T$$

  for any invertible $\boldsymbol{\Gamma}$.

- **dictionary learning**

$$\boldsymbol{Y} = \boldsymbol{A}_0 \boldsymbol{X}_0 = \boldsymbol{A}_0 \boldsymbol{\Pi} \boldsymbol{\Pi}^* \boldsymbol{X}_0$$

  for any signed permutation $\boldsymbol{\Pi}$.

- **blind deconvolution**

$$\boldsymbol{y} = \boldsymbol{a}_0 * \boldsymbol{x}_0 = S_\tau[\boldsymbol{a}_0] * S_{-\tau}[\boldsymbol{x}_0]$$

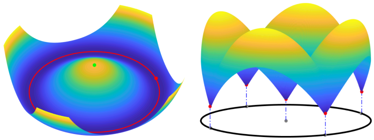
  for any signed shift $\tau$.

# Optimization under Symmetry

**Definition (Symmetric Function)**

Let $\mathbb{G}$ be a group acting on $\mathbb{R}^n$. A function $\varphi : \mathbb{R}^n \to \mathbb{R}^{n'}$ is $\mathbb{G}$-symmetric if for all $\boldsymbol{z} \in \mathbb{R}^n$, $\mathfrak{g} \in \mathbb{G}$, $\varphi(\mathfrak{g} \circ \boldsymbol{z}) = \varphi(\boldsymbol{z})$.

Most symmetric objective functions that arise in structured signal recovery do not have spurious local minimizers or flat saddles.

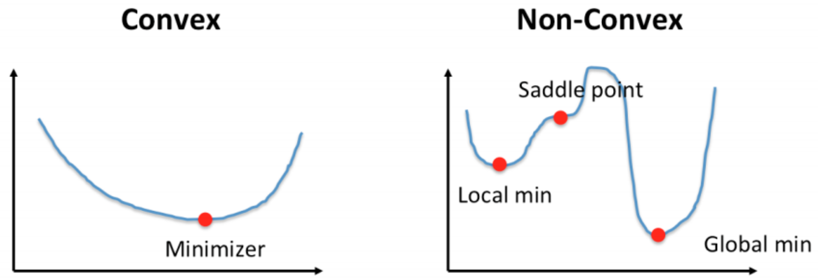


Rotational symmetry    Discrete symmetry

**Slogan 1:** the (only!) local minimizers are symmetric versions of the ground truth.
**Slogan 2:** any local critical point has negative curvature in directions that break symmetry.

# Basic Calculus

Critical points or stationary points: gradient vanishes

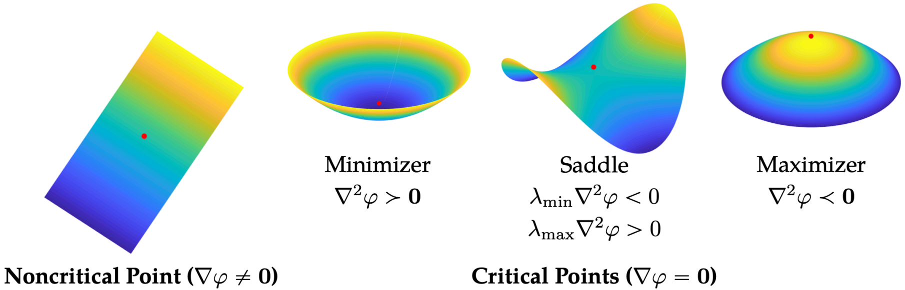



- convex function: critical point = minimizer
- nonconvex function: not all critical points are minimizers

# Basic Calculus

Critical points with non-singular hessian

- minimizer: hessian is positive definite
- saddle points: hessian has both positive and negative eigenvalues
- maximizer: hessian is negative definite



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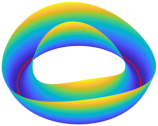
# Problems with Rotational Symmetry

## Nonconvex Problems with Rotational Symmetries

### Eigenspace Computation

*Compute the principal subspace of a symmetric matrix.*



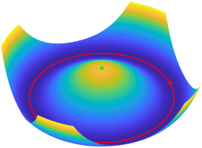
$$\min_{\boldsymbol{X}^*\boldsymbol{X}=\boldsymbol{I}} -\tfrac{1}{2}\operatorname{trace}\left[\boldsymbol{X}^*\boldsymbol{A}\boldsymbol{X}\right].$$

***Symmetry:*** $\boldsymbol{X} \mapsto \boldsymbol{X}\boldsymbol{R}$

$\mathbb{G} = O(r)$

### Generalized Phase Retrieval

*Recover a complex vector $\boldsymbol{x}_0$ from magnitude measurements $\boldsymbol{y} = |\boldsymbol{A}\boldsymbol{x}_0|$.*



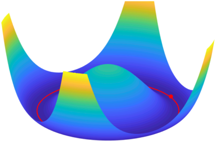
$$\min_{\boldsymbol{x}} \tfrac{1}{2}\|\boldsymbol{y}^2 - |\boldsymbol{A}\boldsymbol{x}|^2\|_2^2.$$

***Symmetry:*** $\boldsymbol{x} \mapsto \boldsymbol{x}e^{i\phi}$

$\mathbb{G} = \mathbb{S}^1 \cong O(2)$

### Matrix Recovery

*Recover a low-rank matrix $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{V}^*$ from incomplete / corrupted observations*



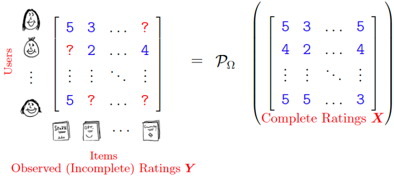
$$\min_{\boldsymbol{U},\boldsymbol{V}} \mathcal{L}(\boldsymbol{Y} - \mathcal{A}[\boldsymbol{U}\boldsymbol{V}^*]) \; + \; \rho(\boldsymbol{U},\boldsymbol{V}).$$

***Symmetry:*** $(\boldsymbol{U},\boldsymbol{V}) \mapsto (\boldsymbol{U}\boldsymbol{\Gamma}, \boldsymbol{V}\boldsymbol{\Gamma}^{-*})$

$\mathbb{G} = \mathrm{GL}(r)$ or $\mathbb{G} = O(r)$

# Low rank matrix recovery

Goal: Given $\boldsymbol{Y} = \mathcal{A}(\boldsymbol{X})$, recover low rank matrix $\boldsymbol{X} = \boldsymbol{U}_0 \boldsymbol{V}_0$




- Convex Formulation

$$\min_{\boldsymbol{X} \in \mathbb{R}^{m \times n}} \quad \|\boldsymbol{X}\|_\star \quad \text{s.t.} \quad \boldsymbol{Y} = \mathcal{A}(\boldsymbol{X})$$

- Nonconvex Formulation

$$\min_{\boldsymbol{U} \in \mathbb{R}^{m \times r}, \boldsymbol{V} \in \mathbb{R}^{n \times r}} \quad \left\|\boldsymbol{Y} - \mathcal{A}(\boldsymbol{U}\boldsymbol{V}^T)\right\|_F^2 + \text{reg}(\boldsymbol{U}, \boldsymbol{V})$$

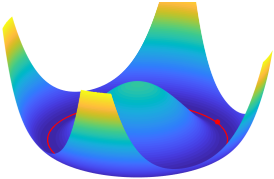
# Low Rank Matrix Recovery

$$\min_{\boldsymbol{U},\boldsymbol{V}} \quad \frac{1}{2}\left\|\boldsymbol{Y} - \mathcal{A}(\boldsymbol{U}\boldsymbol{V}^T)\right\|_F^2 + \mathrm{reg}(\boldsymbol{U},\boldsymbol{V})$$

**Inherent Symmetry**:

$$\boldsymbol{X} = \boldsymbol{U}_0\boldsymbol{V}_0^T = \boldsymbol{U}_0\boldsymbol{\Gamma}\boldsymbol{\Gamma}^{-1}\boldsymbol{V}_0^T$$

for any invertible $\boldsymbol{\Gamma} \in \mathbb{R}^{r\times r}$.



# Low Rank Matrix Recovery

$$\min_{\boldsymbol{U},\boldsymbol{V}} \quad \frac{1}{2}\left\|\boldsymbol{Y} - \mathcal{A}(\boldsymbol{U}\boldsymbol{V}^T)\right\|_F^2 + \text{reg}(\boldsymbol{U},\boldsymbol{V})$$

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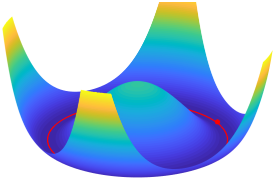


- Are $\left(\boldsymbol{U}_0\boldsymbol{\Gamma}, \boldsymbol{V}_0\boldsymbol{\Gamma}^{-1}\right)$ the only local solutions?
- Does there exist any flat stationary point?

# Rank-1 Symmetric Matrix

Simplifications:

- $\boldsymbol{Y} = \mathcal{A}(\boldsymbol{X}) = \boldsymbol{X}$
- $\boldsymbol{X} = \boldsymbol{U}_0 \boldsymbol{U}_0^T$ is symmetric and rank-1

$$\boldsymbol{X} = \boldsymbol{u}_0 \boldsymbol{u}_0^T = (-\boldsymbol{u}_0)(-\boldsymbol{u}_0^T)$$

the rotational symmetry is reduced to sign symmetry.

Nonconvex formulation:

$$\min_{\boldsymbol{u}} \quad \phi(\boldsymbol{u}) \doteq \frac{1}{4} \left\| \boldsymbol{X} - \boldsymbol{u}\boldsymbol{u}^T \right\|_F^2 + \underbrace{\lambda \left\| \boldsymbol{u} \right\|_2^2}_{const}$$

# Rank-1 Symmetric Matrix

$$\min_{\boldsymbol{u}} \quad \phi(\boldsymbol{u}) \doteq \frac{1}{4} \left\| \boldsymbol{X} - \boldsymbol{u}\boldsymbol{u}^T \right\|_F^2$$

Critical points have zero gradient

$$\begin{aligned} \nabla\phi &= (\boldsymbol{u}\boldsymbol{u}^T - \boldsymbol{X})\boldsymbol{u} \\ &= \|\boldsymbol{u}\|_2^2 \, \boldsymbol{u} - \boldsymbol{X}\boldsymbol{u} \\ &= \mathbf{0} \end{aligned}$$

therefore critical points must be one of the following

- $\boldsymbol{u} = \pm \boldsymbol{u}_0$
- $\boldsymbol{u} = \mathbf{0}$

# Rank-1 Symmetric Matrix

$$\min_{\boldsymbol{u}} \quad \phi(\boldsymbol{u}) \doteq \frac{1}{4} \left\| \boldsymbol{X} - \boldsymbol{u}\boldsymbol{u}^T \right\|_F^2$$

with the second order derivative

$$\nabla^2 \phi = 2\boldsymbol{u}\boldsymbol{u}^T + \|\boldsymbol{u}\|_2^2 \, \boldsymbol{I} - \boldsymbol{X}.$$

# Rank-1 Symmetric Matrix

$$\min_{\boldsymbol{u}} \quad \phi(\boldsymbol{u}) \doteq \frac{1}{4} \left\| \boldsymbol{X} - \boldsymbol{u}\boldsymbol{u}^T \right\|_F^2$$

with the second order derivative

$$\nabla^2 \phi = 2\boldsymbol{u}\boldsymbol{u}^T + \|\boldsymbol{u}\|_2^2 \, \boldsymbol{I} - \boldsymbol{X}.$$

Then the stationary points can be grouped as

- Local minimizer $\boldsymbol{u} = \pm \boldsymbol{u}_0$ and $\boldsymbol{u}\boldsymbol{u}^T = \boldsymbol{X}$

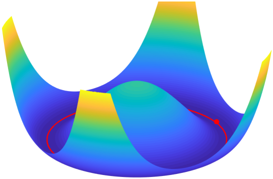
$$\nabla^2 \phi = \boldsymbol{u}\boldsymbol{u}^T + \|\boldsymbol{u}\|_2^2 \, \boldsymbol{I}.$$

- Maximizer $\boldsymbol{u} = \boldsymbol{0}$

$$\nabla^2 \phi = -\boldsymbol{X}.$$

# Low Rank Matrix Recovery

Symmetric low rank matrix



$$\min_{\boldsymbol{U}} \quad \phi(\boldsymbol{u}) \doteq \frac{1}{4} \left\| \boldsymbol{X} - \boldsymbol{U}\boldsymbol{U}^T \right\|_F^2 .$$

General low rank matrix recover

$$\min_{\boldsymbol{U},\boldsymbol{V}} \quad \phi(\boldsymbol{u}) \doteq \frac{1}{2} \left\| \boldsymbol{X} - \boldsymbol{U}\boldsymbol{V}^T \right\|_F^2 + \lambda \left\| \boldsymbol{U} \right\|_F^2 + \lambda \left\| \boldsymbol{V} \right\|_F^2 .$$

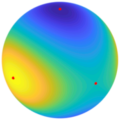
**Local minimizers:** *are ground truth* $\boldsymbol{U}_0$ *and* $\boldsymbol{V}_0$ *up to rotation;*
**Negative curvature:** *between multiple local minimizers.*

# Problems with Discrete Symmetry

## Nonconvex Problems with Discrete Symmetries

### Eigenvector Computation

*Maximize a quadratic form over the sphere.*



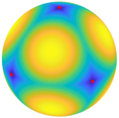
$\max_{\boldsymbol{x}\in\mathbb{S}^{n-1}} \frac{1}{2}\boldsymbol{x}^*\boldsymbol{A}\boldsymbol{x}.$

*Symmetry:* $\boldsymbol{x} \mapsto -\boldsymbol{x}$

$\mathbb{G} = \{\pm 1\}$

### Dictionary Learning

*Approximate a given matrix* $\boldsymbol{Y}$ *as* $\boldsymbol{Y} \approx \boldsymbol{A}\boldsymbol{X}$*, with* $\boldsymbol{X}$ *sparse*



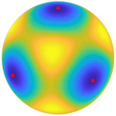
$\min_{\boldsymbol{A}\in\mathcal{A},\boldsymbol{X}} \frac{1}{2}\|\boldsymbol{Y}-\boldsymbol{A}\boldsymbol{X}\|_F^2 + \lambda\|\boldsymbol{X}\|_1.$

*Symmetry:* $(\boldsymbol{A},\boldsymbol{X}) \mapsto (\boldsymbol{A}\boldsymbol{\Gamma}, \boldsymbol{X}\boldsymbol{\Gamma}^*)$

$\mathbb{G} = \mathrm{SP}(n)$

### Tensor Decomposition

*Determine components* $\boldsymbol{a}_i$ *of an orthogonal decomposable tensor* $\boldsymbol{T} = \sum_i \boldsymbol{a}_i \otimes \boldsymbol{a}_i \otimes \boldsymbol{a}_i \otimes \boldsymbol{a}_i$



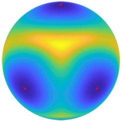
$\max_{\boldsymbol{X}\in O(n)} \sum_i \boldsymbol{T}(\boldsymbol{x}_i,\boldsymbol{x}_i,\boldsymbol{x}_i,\boldsymbol{x}_i).$

*Symmetry:* $\boldsymbol{X} \mapsto \boldsymbol{X}\boldsymbol{\Gamma}$

$\mathbb{G} = \mathrm{P}(n)$

### Short-and-Sparse Deconvolution

*Recover a short* $\boldsymbol{a}$ *and a sparse* $\boldsymbol{x}$ *from their convolution* $\boldsymbol{y} = \boldsymbol{a} * \boldsymbol{x}$.



$\min_{\boldsymbol{a},\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{a}*\boldsymbol{x}\|_2^2 + \lambda\|\boldsymbol{x}\|_1.$

*Symmetry:* $(\boldsymbol{a},\boldsymbol{x}) \mapsto (\alpha s_\tau[\boldsymbol{a}], \alpha^{-1}s_{-\tau}[\boldsymbol{x}])$

$\mathbb{G} = \mathbb{Z}_n \times \mathbb{R}_* \text{ or } \mathbb{G} = \mathbb{Z}_n \times \{\pm 1\}$

# Dictionary Learning

Goal: Given dataset $\boldsymbol{Y}$, find the optimal dictionary $\boldsymbol{A}$ that renders the sparsest coefficient $\boldsymbol{X}$

$$\min_{\boldsymbol{A},\boldsymbol{X}} \quad \|\boldsymbol{X}\|_1 \quad \text{s.t.} \quad \boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}.$$

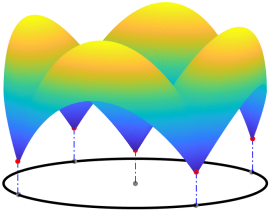
In presence of noise, the optimization problem can be rewritten as

$$\min_{\boldsymbol{A},\boldsymbol{X}} \quad \frac{1}{2}\|\boldsymbol{Y} - \boldsymbol{A}\boldsymbol{X}\|_F^2 + \lambda \|\boldsymbol{X}\|_1 .$$

**Inherent Symmetry**:

$$\boldsymbol{Y} = \boldsymbol{A}_0 \boldsymbol{\Gamma}\boldsymbol{\Gamma}^* \boldsymbol{X}_0,$$

for any signed permutation matrix $\boldsymbol{\Gamma}$.



# Orthogonal Dictionary Learning

- Input: matrix $\boldsymbol{Y}$ which is the product of an orthogonal matrix $\boldsymbol{A}_0$ (called a dictionary) and a sparse matrix $\boldsymbol{X}_0$:

$$\boldsymbol{Y} = \boldsymbol{A}_0 \boldsymbol{X}_0, \quad \boldsymbol{A}_0 \boldsymbol{A}_0^* = \boldsymbol{I}, \boldsymbol{X}_0 \text{ sparse.}$$

- Optimization Formulation

$$\min_{\boldsymbol{A}, \boldsymbol{X}} \quad \|\boldsymbol{X}\|_1 \quad \text{s.t.} \quad \boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}, \quad \boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{I}.$$

- Given the optimization constraint, $\boldsymbol{X}$ is uniquely defined in terms of $\boldsymbol{A}$

$$\boldsymbol{X} = \boldsymbol{A}^* \boldsymbol{A} \boldsymbol{X} = \boldsymbol{A}^* \boldsymbol{Y}.$$

- Equivalent formulation

$$\min_{\boldsymbol{A} \in \mathcal{O}(n)} \quad \|\boldsymbol{A}^* \boldsymbol{Y}\|_1 .$$

# Orthogonal Dictionary Learning

Instead of aiming to solve the entire matrix $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]$ at once via

$$\min_{\boldsymbol{A} \in \mathcal{O}(n)} \quad \|\boldsymbol{A}^* \boldsymbol{Y}\|_1 .$$

A simpler model problem solves for the columns $\boldsymbol{a}_i$ one at a time

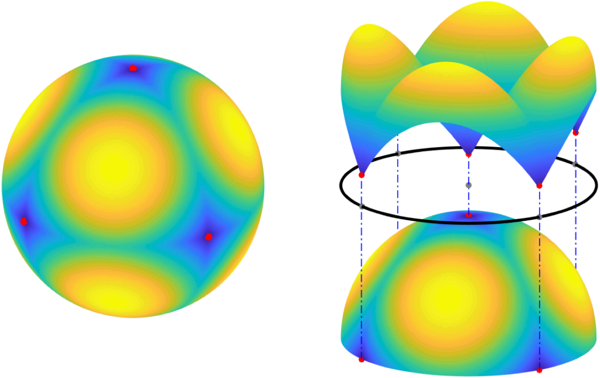
$$\min_{\|\boldsymbol{a}\|_2 = 1} \quad \|\boldsymbol{a}^* \boldsymbol{Y}\|_1 .$$

Stationary Points:

- $\boldsymbol{a} = \pm \boldsymbol{a}_i$, then the Hessian is positive definite
- $\boldsymbol{a} = \sum_{i \in I} \pm \frac{1}{\sqrt{|I|}} \boldsymbol{a}_i$, there exist negative curvatures alone $\boldsymbol{a}_i (i \in I)$

# Orthogonal Dictionary Learning — Geometry

**Local minimizers** are ground truth $\boldsymbol{a}_i$ or $-\boldsymbol{a}_i$.
**Negative curvature** between multiple local minimizers.

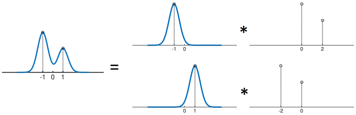


# Short-and-Sparse Blind Deconvolution

Goal: Given convolutional data $\boldsymbol{y}$, find the short signal $\boldsymbol{a}$ and the sparse signal $\boldsymbol{x}$ such that $\boldsymbol{y} = \boldsymbol{a} * \boldsymbol{x}$.

**Inherent Symmetry**:

$$\boldsymbol{y} = \boldsymbol{a}_0 * \boldsymbol{x}_0 = \alpha s_l[\boldsymbol{a}_0] * \frac{1}{\alpha} s_{-l}[\boldsymbol{x}_0]$$

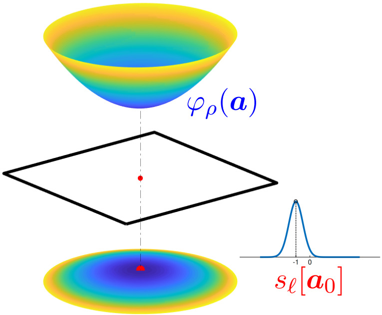



for any shift $l$ and nonzero scaling.

The practical optimization problem can be written as

$$\min_{\|a\|_F^2 = 1, \boldsymbol{x}} \quad \frac{1}{2} \|\boldsymbol{y} - \boldsymbol{a} * \boldsymbol{x}\|_F^2 + \lambda \|\boldsymbol{x}\|_1 .$$

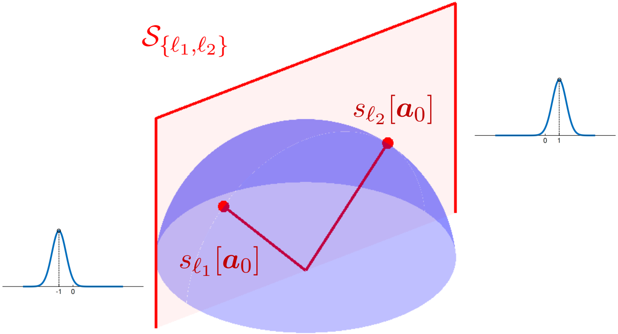
# Objective Function – Near One Shift




$$\mathbb{S}^{p-1} \cap \{\boldsymbol{a} \in \mathbb{S}^{p-1} \mid \|\boldsymbol{a} - s_\ell[\boldsymbol{a}_0]\|_2 \le r\}$$

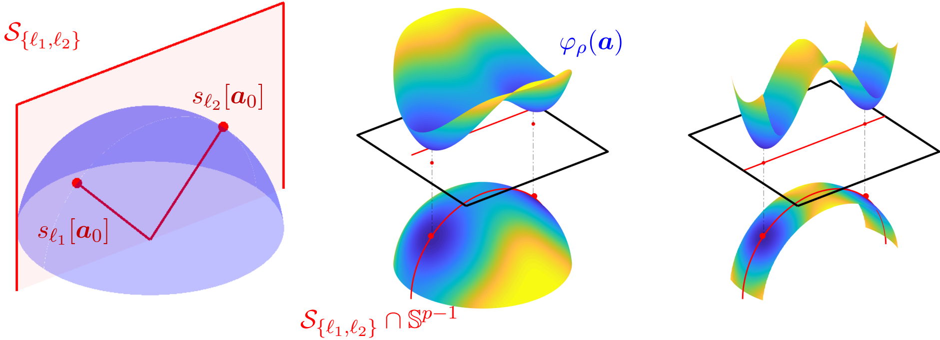
Objective function is **strongly convex** near a shift $s_\ell[\boldsymbol{a}_0]$ of the ground truth.

# Objective Function – Linear Span of Two Shifts



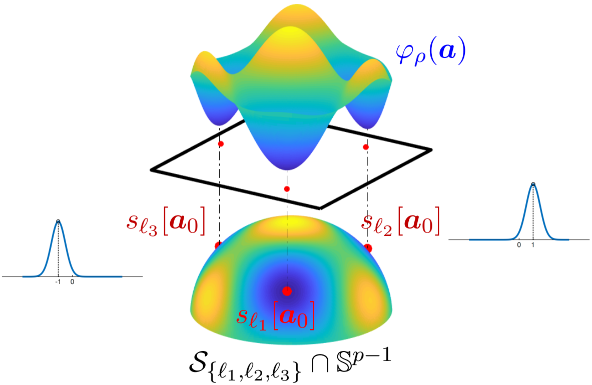

**Subspace** $\mathcal{S}_{\{\ell_1,\ell_2\}} = \{\alpha_{\ell_1} s_{\ell_1}[\boldsymbol{a}_0] + \alpha_{\ell_2} s_{\ell_2}[\boldsymbol{a}_0] \mid \alpha_{\ell_1}, \alpha_{\ell_2} \in \mathbb{R}\}$.

# Objective Function – Linear Span of Two Shifts




**Local minimizers** are near signed shifts $\pm s_\ell[\boldsymbol{a}_0]$.
**Negative curvature** between two shifts $s_{\ell_1}[\boldsymbol{a}_0]$, $s_{\ell_2}[\boldsymbol{a}_0]$.

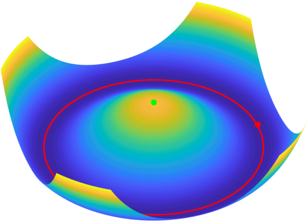
# Objective Function – Multiple Shifts



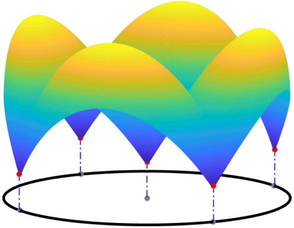

Objective $\varphi_\rho$ over the linear span $\mathcal{S}_{\ell_1,\ell_2,\ell_3} = \{\sum_{i=1}^{3} \alpha_{\ell_i} s_{\ell_i}[\boldsymbol{a}_0]\}$
**Local minimizers** are near signed shifts $\pm s_{\ell_i}[\boldsymbol{a}_0]$.

# Symmetry and Nonconvexity

- the (only!) local minimizers are symmetric versions of the ground truth.
- there is negative curvature in directions that break symmetry.



**Rotational symmetry**



**Discrete symmetry**

# Outline

1 Introduction & Motivation of Nonconvex Optimization
Motivating Examples
Nonlinearality, Nonconvexity, and Symmetry

2 Symmetry & Geometry for Nonconvex Problems in Practice
Problems with Rotational Symmetry
Problems with Discrete Symmetry

3 Efficient Nonconvex Optimization
Objectives of Nonconvex Optimization

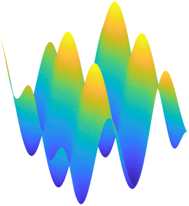
# Nonconvex Optimization

Consider the problem of
minimizing a general nonconvex function:

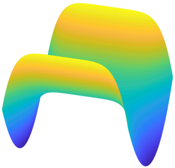
$$\min_{\boldsymbol{x}} f(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathsf{C}. \tag{5}$$

In the worst case, even finding
a *local* minimizer can be NP-hard[2].

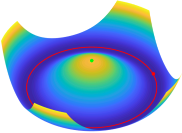
Nonconvex problems that
arise from natural physical, geometrical,
or statistical origins typically have
nice structures, in terms of symmetries!



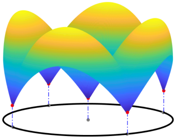
**Spurious local minimizers**



**Flat saddle points**



**Rotational symmetry**



**Discrete symmetry**

[2] Some NP-complete problems in quadratic and nonlinear programming, K.G Murty and S. N. Kabadi, 1987

# Objectives

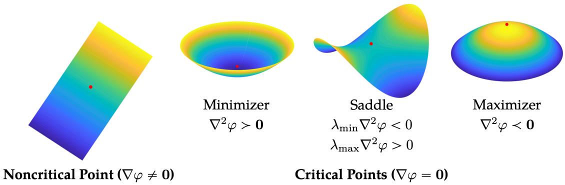
Hence typically people seek to work with relatively benign (gradient/Hessian Lipschitz continuous) functions:

$$\forall \boldsymbol{x}, \boldsymbol{y} \quad \|\nabla f(\boldsymbol{y}) - \nabla f(\boldsymbol{x})\|_2 \le L_1 \|\boldsymbol{y} - \boldsymbol{x}\|_2 \tag{6}$$

with benign objectives:

1. convergence to some critical point $\boldsymbol{x}_\star$ such that: $\nabla f(\boldsymbol{x}_\star) = \mathbf{0}$;
2. the critical point $\boldsymbol{x}_\star$ is second-order stationary: $\nabla^2 f(\boldsymbol{x}_\star) \succeq \mathbf{0}$.

**Example:** a function $\varphi$ with symmetry only has **regular** critical points, while general $f$ could have irregular second-order stationary points:

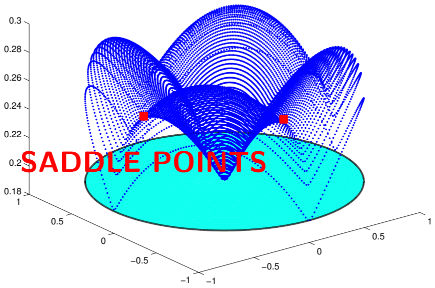


# “Any Reasonable Algorithm” Works

**Key issue**: using negative curvature

$$\lambda_{\min}(\mathrm{Hess} f) < 0$$

to escape saddles.



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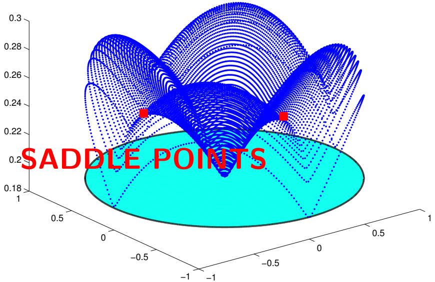


**Efficient (polynomial time) methods**:
Trust region method, analyses in [Sun, Qu, W., '17]
Curvilinear search, [Goldfarb, Mu, W., Zhou, '16]
Noisy (stochastic) gradient descent, [Jin et. al. '17].

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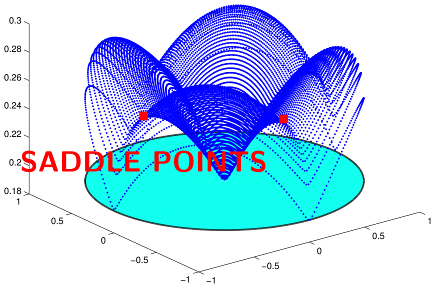


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Curvilinear search, [Goldfarb, Mu, W., Zhou, '16]

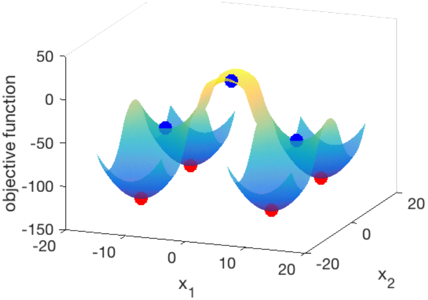
Noisy (stochastic) gradient descent, [Jin et. al. '17].

**Randomly initialized gradient descent** ....

Obtains a minimizer almost surely [Lee et. al. '16].

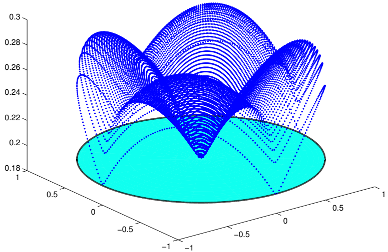
Efficient for matrix completion, dictionary learning, . . . not efficient in general.

# Worst Case vs. Naturally Occurring Strict Saddle Functions




**Worst Case**

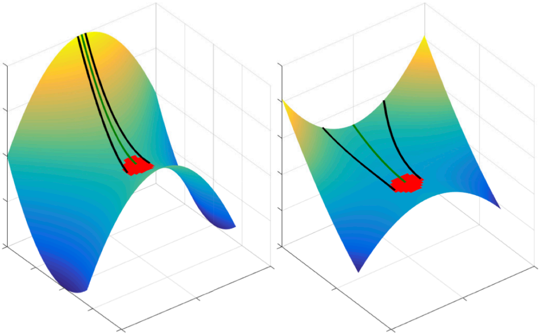
[Du, Jin, Lee, Jordan, Poczos, Singh '17]
Concentration around stable manifold




**Naturally Occuring**

DL, Other sparsification problems
Dispersion away from stable manifold

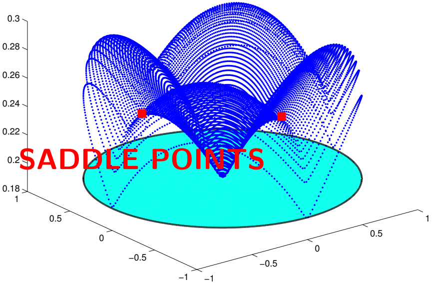
# Worst Case vs. Naturally Occurring Strict Saddle Functions

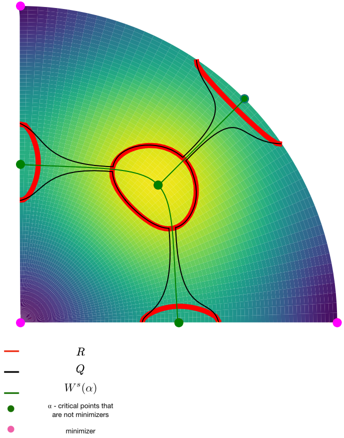


- Red: "slow region" of small gradient around a saddle point.
- Green: stable manifold associated with the saddle point.
- Black: points that flow to the slow region.

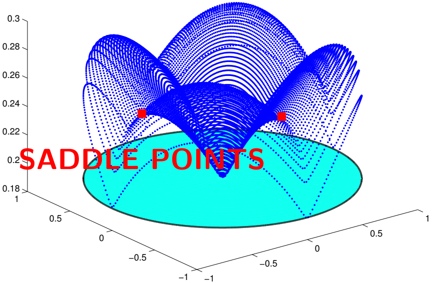
- Left: global negative curvature normal to the stable manifold
- Right: positive curvature normal to the stable manifold – randomly initialized gradient descent is more likely to encounter the slow region.

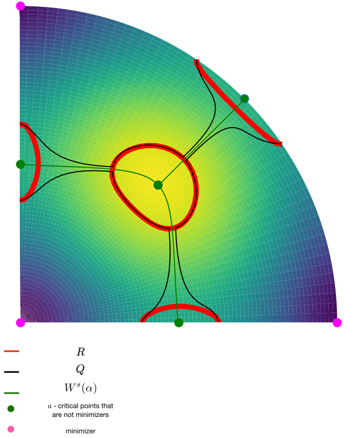
# Gradient Descent Works for DL and Related Problems



# Gradient Descent Works for DL and Related Problems




**Dispersive structure**: Negative curvature $\perp$ stable manifolds.

W.h.p. in random initialization $\boldsymbol{q}^{(0)} \sim \mathrm{uni}(\mathbb{S}^{n-1})$, **convergence to a neighborhood of a minimizer in polynomial iterations.** [Gilboa,

# Conclusion and Coming Attractions

For Nonconvex, Sparse and Low-rank problems

- **Benign Geometry**:
  - The only local minimizers are symmetric copies of the ground truth
  - There exist negative curvatures breaking symmetry
- **Efficient Algorithms**:
  - gradient descent algorithms always suffice
  - proximal, projection, acceleration steps can be transferred over

**Thank You! Questions?**